

Kleiner's theorem for unitary representations of posets

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Abstract

A subspace representation of a poset $\mathcal{S} = \{s_1, \dots, s_t\}$ is given by a system $(V; V_1, \dots, V_t)$ consisting of a vector space V and its subspaces V_i such that $V_i \subseteq V_j$ if $s_i \prec s_j$. For each real-valued vector $\chi = (\chi_1, \dots, \chi_t)$ with positive components, we define a unitary χ -representation of \mathcal{S} as a system $(U; U_1, \dots, U_t)$ that consists of a unitary space U and its subspaces U_i such that $U_i \subseteq U_j$ if $s_i \prec s_j$ and satisfies $\chi_1 P_1 + \dots + \chi_t P_t = 1$, in which P_i is the orthogonal projection onto U_i .

We prove that \mathcal{S} has a finite number of unitarily nonequivalent indecomposable χ -representations for each weight χ if and only if \mathcal{S} has a finite number of nonequivalent indecomposable subspace representations; that is, if and only if \mathcal{S} contains any of Kleiner's critical posets.

Keywords: Representations of partially ordered sets, Representation-finite type, Kleiner's theorem

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1. Introduction

Kleiner [6] described all partially ordered sets (*posets*) with finite number of nonequivalent indecomposable representations. We extend his description to unitary representations of posets.

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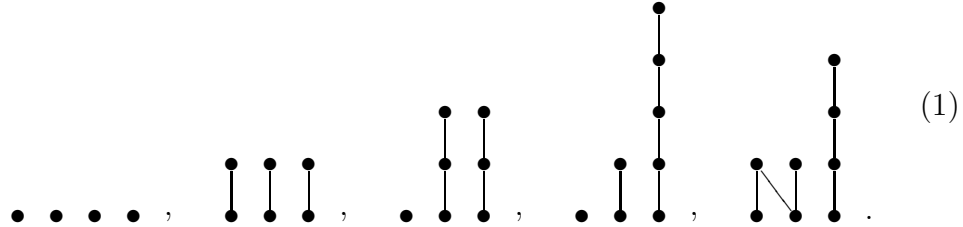
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The notion of poset representations was introduced by Nazarova and Roiter [11] (see also [2, 14]). A *matrix representation* of a finite poset $\mathcal{S} = \{s_1, \dots, s_t\}$ over a field \mathbb{F} is a block matrix $\mathcal{A} = [A_1 | \dots | A_t]$ over \mathbb{F} . Two representations $\mathcal{A} = [A_1 | \dots | A_t]$ and $\mathcal{B} = [B_1 | \dots | B_t]$ are *equivalent* if \mathcal{A} can be reduced to \mathcal{B} by elementary row transformations, elementary column transformations within A_i , and additions of linear combinations of columns of A_i to columns of A_j if $s_i \prec s_j$. The *direct sum* of \mathcal{A} and \mathcal{B} is the representation

$$\mathcal{A} \oplus \mathcal{B} := \left[\begin{array}{cc|cc|ccc|cc} A_1 & 0 & A_2 & 0 & \dots & A_t & 0 \\ 0 & B_1 & 0 & B_2 & \dots & 0 & B_t \end{array} \right].$$

A representation is called *indecomposable* if it is not equivalent to a direct sum of two representations. It is sufficient to classify only indecomposable representations since each representation is equivalent to a direct sum of indecomposable representations, uniquely determined up to isomorphism of summands.

Kleiner [6] (see also [2, Theorem 5.1] and [14, Theorem 10.1]) proved that a poset \mathcal{S} has only a finite number of nonequivalent indecomposable representations if and only if it does not contain a full poset whose Hasse diagram is one of the form



An equivalent definition of poset representations can be given in terms of subspaces. A *subspace representation* of $\mathcal{S} = \{s_1, \dots, s_t\}$ is a tuple $\mathcal{V} = (V; V_1, \dots, V_t)$, in which V is a vector space over \mathbb{F} and V_1, \dots, V_t are its subspaces such that $V_i \subseteq V_j$ if $s_i \prec s_j$ (i.e., each representation is a homomorphism from \mathcal{S} to the poset of all subspaces of V). Two subspace representations $\mathcal{V} = (V; V_1, \dots, V_t)$ and $\mathcal{W} = (W; W_1, \dots, W_t)$ are *equivalent* if there exists a linear bijection $g : V \rightarrow W$ such that $g(V_i) = W_i$ for all i . For each subspace representation $\mathcal{V} = (V; V_1, \dots, V_t)$, one can construct a matrix representation $\mathcal{A} = [A_1 | \dots | A_t]$ in such a way that (i) for each i the

columns of all A_j with $s_j \preceq s_i$ generate the subspace V_i ; and (ii) two subspace representations are equivalent if and only if the corresponding matrix representations are equivalent; see [14, Chapter 3].

From now on, all representations that we consider are over the field \mathbb{C} of complex numbers. By a *unitary representation of dimension d* , we mean a subspace representation $\mathcal{U} = (U; U_1, \dots, U_t)$ in which U is a unitary space of dimension d . Two unitary representations $\mathcal{U} = (U; U_1, \dots, U_t)$ and $\mathcal{V} = (V; V_1, \dots, V_t)$ of a poset \mathcal{S} are *unitarily equivalent* if there exists a unitary bijection $\varphi : U \rightarrow V$ such that $\varphi(U_i) = V_i$ for all i . The *orthogonal sum* of unitary representations \mathcal{U} and \mathcal{V} is the unitary representation

$$\mathcal{U} \perp \mathcal{V} := (U \perp V; U_1 \perp V_1, \dots, U_t \perp V_t),$$

in which $U \perp V$ denotes the orthogonal sum of U and V . A unitary representation is called *orthogonally indecomposable* if it is not equivalent to an orthogonal sum of two unitary representations.

Note that the problem of classifying unitary representations is hopeless even for the poset $\mathcal{S} = \{s_1, s_2, s_3 \mid s_1 \prec s_2\}$ since by [10, Theorem 4] it contains the problem of classifying an operator on a unitary space, and hence it contains the problem of classifying any system of operators on unitary spaces [10, 13]. The classification becomes possible for a broader class of posets if we impose additional conditions on unitary representations.

We denote the orthogonal projection onto a subspace $M \subset U$ by P_M and the set of positive real numbers by \mathbb{R}_+ . We say that a unitary representation $\mathcal{U} = (U; U_1, \dots, U_t)$ is a *representation of weight $\chi = (\chi_1, \dots, \chi_t) \in \mathbb{R}_+^t$* (or χ -representation) if

$$\chi_1 P_{U_1} + \dots + \chi_t P_{U_t} = 1; \tag{2}$$

such relations appear in many areas of mathematics, see for example [1, 7, 9, 16, 17] and references therein.

Our goal is to prove that Kleiner's theorem holds for χ -representations too:

Theorem 1. *The following conditions are equivalent for each finite poset \mathcal{S} with t elements:*

- (i) *For each $\chi \in \mathbb{R}_+^t$, \mathcal{S} has only a finite number of indecomposable unitarily nonequivalent χ -representations;*
- (ii) *For each $\chi \in \mathbb{R}_+^t$ and $d \in \mathbb{N}$, \mathcal{S} has only a finite number of indecomposable unitarily nonequivalent χ -representations of dimension d ;*

- (iii) \mathcal{S} does not contain a full poset whose Hasse diagram is one of the form (1).

2. Preliminaries

In what follows we suppose that the elements of a poset \mathcal{S} are numbered from 1 to $|\mathcal{S}|$. A poset is called *primitive* and is denoted by (t_1, \dots, t_s) if it is the disjoint (cardinal) sum of linearly ordered sets of orders t_i . The diagrams (1) and corresponding posets are called *critical*. The poset which corresponds to the last diagram in the list (1) is denoted by $(N, 4)$. To simplify the notation we denote a subspace representation $(V; V_1, \dots, V_t)$ of \mathcal{S} by $(V; V_i)_{i \in \mathcal{S}}$. The similar notation will be used for unitary representations and weights.

A subspace representation $\mathcal{V} = (V; V_i)_{i \in \mathcal{S}}$ is called *schurian* if all its endomorphisms are trivial; that is, the ring $\text{End}(\mathcal{V}) := \{g \in M_{\dim V}(\mathbb{F}) \mid g(V_i) \subseteq V_i, i \in \mathcal{S}\}$ is isomorphic to \mathbb{F} . Any schurian representation is indecomposable.

Any unitary representation $\mathcal{U} = (U; U_i)_{i \in \mathcal{P}}$ can be viewed as a subspace representation; the forgetful map is denoted by F . If \mathcal{U} is an indecomposable χ -representation, then $F(\mathcal{U})$ is schurian (see [9, Theorem 1]).

Lemma 2. *Let $P_i, Q_i \in M_n(\mathbb{C})$, $i = 1, \dots, m$ be orthogonal projections such that*

$$\chi_1 P_1 + \dots + \chi_m P_m = \chi_1 Q_1 + \dots + \chi_m Q_m \quad (3)$$

for (χ_1, \dots, χ_m) with positive real χ_i . Let there exist a diagonal matrix $D = \text{diag}(r_1, \dots, r_n)$ with positive components such that $P_i D Q_i = D Q_i$ for all i . Then $r_1 = \dots = r_n$ and $P_i = Q_i$ for all i .

Proof. Write $P_i = [p_{k,l}^{(i)}]$, $Q_i = [q_{k,l}^{(i)}]$, $P_i D Q_i = [t_{k,l}^{(i)}]$, where $t_{k,l}^{(i)} = \sum_{j=1}^n r_j p_{k,j}^{(i)} \overline{q_{l,j}^{(i)}}$. Without losing generality, we may assume that $r_1 = \max\{r_1, \dots, r_n\}$. Since $D \sum_{i=1}^m \chi_i Q_i = \sum_{i=1}^m \chi_i P_i D Q_i$, we have

$$r_1 \sum_{i=1}^m \chi_i q_{1,1}^{(i)} = \sum_{i=1}^m \chi_i \sum_{j=1}^n r_j p_{1,j}^{(i)} \overline{q_{1,j}^{(i)}} \leq r_1 \left| \sum_{i=1}^m \chi_i \sum_{j=1}^n p_{1,j}^{(i)} \overline{q_{1,j}^{(i)}} \right|. \quad (4)$$

For

$$\begin{aligned} x &:= [\sqrt{\chi_1} p_{1,1}^{(1)}, \dots, \sqrt{\chi_1} p_{1,n}^{(1)}, \dots, \sqrt{\chi_m} p_{1,1}^{(m)}, \dots, \sqrt{\chi_m} p_{1,n}^{(m)}]^T \in \mathbb{C}^{nm}, \\ y &:= [\sqrt{\chi_1} q_{1,1}^{(1)}, \dots, \sqrt{\chi_1} q_{1,n}^{(1)}, \dots, \sqrt{\chi_m} q_{1,1}^{(m)}, \dots, \sqrt{\chi_m} q_{1,n}^{(m)}]^T \in \mathbb{C}^{nm}, \end{aligned}$$

we have

$$(x, y) = \sum_{i=1}^{nm} x_i \bar{y}_i = \sum_{i=1}^m \chi_i \sum_{j=1}^n p_{1,j}^{(i)} \overline{q_{1,j}^{(i)}}.$$

Note that

$$|x|^2 = \sum_{i=1}^{nm} x_i \bar{x}_i = \sum_{i=1}^m \chi_i \sum_{j=1}^n p_{1,j}^{(i)} \overline{p_{1,j}^{(i)}} = \sum_{i=1}^m \chi_i p_{1,1}^{(i)}.$$

Similarly, $|y|^2 = \sum_{i=1}^m \chi_i q_{1,1}^{(i)}$. By (3), we have $\sum_{i=1}^m \chi_i p_{1,1}^{(i)} = \sum_{i=1}^m \chi_i q_{1,1}^{(i)}$, hence $|x| = |y|$. By the Cauchy-Schwartz inequality,

$$\left| \sum_{i=1}^m \chi_i \sum_{j=1}^n p_{1,j}^{(i)} \overline{q_{1,j}^{(i)}} \right| \leq \sqrt{\sum_{i=1}^m \chi_i p_{1,1}^{(i)}} \sqrt{\sum_{i=1}^m \chi_i q_{1,1}^{(i)}} = \sum_{i=1}^m \chi_i q_{1,1}^{(i)}.$$

Comparing this inequality with (4), we have $r_1 = \dots = r_n$ and $P_i = Q_i$ for all i . \square

Theorem 3. *Two χ -representations $\mathcal{U} = (U; U_i)_{i \in \mathcal{S}}$ and $\mathcal{U}' = (U'; U'_i)_{i \in \mathcal{S}}$ are unitarily equivalent if and only if the corresponding subspace representations $F(\mathcal{U})$ and $F(\mathcal{U}')$ are equivalent.*

Proof. If \mathcal{U} is unitarily equivalent to \mathcal{U}' , then $F(\mathcal{U})$ is equivalent to $F(\mathcal{U}')$. Let us prove the converse statement. $F(\mathcal{U})$ is equivalent to $F(\mathcal{U}')$ if and only if there exists an invertible $g : U \rightarrow U'$ such that

$$g^{-1} P_{U'_i} g P_{U_i} = P_{U_i}, \quad g P_{U_i} g^{-1} P_{U'_i} = P_{U'_i}, \quad i \in \mathcal{S}.$$

Let $g = \varphi \psi D \psi^*$ be the polar decomposition of g , where $\varphi : U \rightarrow U'$ and $\psi : U \rightarrow U$ are unitary maps and D is a positively defined diagonal operator. Then

$$(\psi D^{-1} \psi^* \varphi^*) P_{U'_i} (\varphi \psi D \psi^*) P_{U_i} = P_{U_i}, \quad i \in \mathcal{S}.$$

Hence $(\psi^* \varphi^* P_{U'_i} \varphi \psi) D (\psi^* P_{U_i} \psi) = D (\psi^* P_{U_i} \psi)$ for all i . Since \mathcal{U} and \mathcal{U}' are χ -representations,

$$\sum_{i \in \mathcal{S}} \chi_i (\psi^* \varphi^* P_{U'_i} \varphi \psi) = \chi_0 I, \quad \sum_{i \in \mathcal{S}} \chi_i (\psi^* P_{U_i} \psi) = \chi_0 I.$$

Lemma 2 ensures that $\psi^* \varphi^* P_{U'_i} \varphi \psi = \psi^* P_{U_i} \psi$ for all i . Therefore, $\varphi^* P_{U'_i} \varphi = P_{U_i}$ for all i , and so \mathcal{U} is unitarily equivalent to \mathcal{U}' . \square

Remark 4. By similar argumentation, one can show that χ -representation \mathcal{U} is orthogonally indecomposable if and only if $F(\mathcal{U})$ is indecomposable. The connection between usual and orthoscalar representations of quivers was established [9, Theorem 1] in the same way as in Theorem 3.

A representation $\mathcal{V} = (V; V_i)_{i \in \mathcal{S}}$ of weight $\chi = (\chi_i)_{i \in \mathcal{S}}$ is called χ -stable if $\sum_{i \in \mathcal{S}} \chi_i \dim V_i = \dim V$ and

$$\sum_{i \in \mathcal{S}} \chi_i \dim(V_i \cap M) < \dim M$$

for any proper subspace $0 \neq M \subset V$.

Lemma 5. *If $\mathcal{U} = (U; U_i)_{i \in \mathcal{S}}$ is an indecomposable χ -representation, then $F(\mathcal{U})$ is χ -stable.*

Proof. Equating the traces of both sides in (2), we obtain $\sum_{i \in \mathcal{S}} \chi_i \dim U_i = \dim U$. If M is any proper subspace of U , then $\sum_{i \in \mathcal{S}} \chi_i P_{U_i} P_M = P_M$. Equating the traces of both sides in the last equality, we get

$$\sum_{i \in \mathcal{S}} \chi_i \operatorname{tr}(P_{U_i} P_M) = \dim M.$$

By [4, Theorem 2], $\operatorname{tr}(P_{M_1 \cap M_2}) \leq \operatorname{tr}(P_{M_1} P_{M_2})$ for each two subspaces M_1 and M_2 , and so

$$\sum_{i \in \mathcal{S}} \chi_i \operatorname{tr}(P_{U_i \cap M}) \leq \sum_{i \in \mathcal{S}} \chi_i \operatorname{tr}(P_{U_i} P_M) = \dim M.$$

It remains to prove that the last inequality is strict. Indeed, assume that $\operatorname{tr}(P_{U_i \cap M}) = \operatorname{tr}(P_{U_i} P_M)$ for all i . Then each P_{U_i} commutes with P_M . Hence the subspace M is invariant with respect to the projections P_{U_i} and the representation \mathcal{U} is decomposable. This contradicts the assumption. \square

The converse statement to Lemma 5 also holds: if a representation $\mathcal{V} = (V; V_i)_{i \in \mathcal{S}}$ is χ -stable, then one can choose a scalar product in V in such a way that \mathcal{V} becomes a χ -representation; see [5, Theorem 3.5]. Using results from [5, 7, 16], one can prove the following theorem.

Theorem 6. *An indecomposable unitary representation \mathcal{U} is a χ -representation if and only if the corresponding subspace representation $F(\mathcal{U})$ is χ -stable.*

3. Proof of Theorem 1

The implication (i) \Rightarrow (ii) is trivial.

(iii) \Rightarrow (i). Assume that the Hasse diagram of \mathcal{S} does not contain any of critical diagrams (1). If \mathcal{S} has an infinite number of indecomposable unitarily nonequivalent χ -representations for some weight χ , then by Theorem 3 it has an infinite number of nonequivalent indecomposable subspace representations. By Kleiner's theorem, \mathcal{S} contains a critical diagram; a contradiction.

(ii) \Rightarrow (iii). We say that a poset \mathcal{S} is *unitary representation-infinite* if there exist $d \in \mathbb{N}$ and $\chi^{\mathcal{S}} \in \mathbb{R}_+^{|\mathcal{S}|}$ such that \mathcal{S} has an infinite number of indecomposable unitarily nonequivalent $\chi^{\mathcal{S}}$ -representations of dimension d . Our aim is to prove that critical posets are unitary representation-infinite.

One can show that critical primitive posets are unitary representation-infinite using [1, 8, 12]. Namely, there exists a correspondence between the χ -representations of a given poset \mathcal{S} and the representations of a certain $*$ -algebra $\mathcal{A}_{\Gamma, \omega}$ associated with a star-shaped graph Γ , which is determined by the Hasse diagram of \mathcal{S} , and the parameter ω is determined by the weight χ . If Γ is an extended Dynkin graph (which corresponds to some primitive critical \mathcal{S}), then one can choose the parameter ω such that $\mathcal{A}_{\Gamma, \omega}$ has an infinite number of unitarily nonequivalent irreducible representations. The complete description of such representations was given in [1, 8, 12] (see also Remark 11). But we use another method that handles both primitive and non-primitive cases.

Denote by $e_i^{(n)}$ (or e_i if no confusion can arise) the n -dimensional vector in which the i -th coordinate is 1 and the others are 0. Denote by $e_{i_1 \dots i_k}$ the vector $e_{i_1} + \dots + e_{i_k}$ and by $\langle x_1, \dots, x_m \rangle$ the vector space spanned by $x_1, \dots, x_m \in \mathbb{C}^n$.

For each critical poset \mathcal{S} , we define a family of its subspace representations $\mathcal{V}_\lambda(\mathcal{S})$ that depend on a complex parameter $\lambda \in \mathbb{C}$.

- If $\mathcal{S} = (1, 1, 1, 1)$, then $\mathcal{V}_\lambda(\mathcal{S})$ consists of the space \mathbb{C}^2 and its subspaces

$$\langle e_1 \rangle \quad \langle e_2 \rangle \quad \langle e_1 + e_2 \rangle \quad \langle e_1 + \lambda e_2 \rangle$$

- If $\mathcal{S} = (2, 2, 2)$, then $\mathcal{V}_\lambda(\mathcal{S})$ consists of the space \mathbb{C}^3 and its subspaces

$$\begin{array}{ccccc} \langle e_{123}, e_1 + \lambda e_3 \rangle & \langle e_1, e_2 \rangle & \langle e_2, e_3 \rangle \\ \uparrow & \uparrow & \uparrow \\ \langle e_{123} \rangle & \langle e_1 \rangle & \langle e_3 \rangle \end{array}$$

- If $\mathcal{S} = (1, 3, 3)$, then $\mathcal{V}_\lambda(\mathcal{S})$ consists of the space \mathbb{C}^4 and its subspaces

$$\begin{array}{ccccc}
 & \langle e_1, e_4, e_2 + \lambda e_3 \rangle & & \langle e_1, e_2, e_3 \rangle & \\
 & \uparrow & & \uparrow & \\
 & \langle e_1, e_4 \rangle & & \langle e_2, e_3 \rangle & \\
 & \uparrow & & \uparrow & \\
 \langle e_{123}, e_{24} \rangle & \langle e_4 \rangle & & \langle e_3 \rangle &
 \end{array}$$

- If $\mathcal{S} = (1, 2, 5)$, then $\mathcal{V}_\lambda(\mathcal{S})$ consists of the space \mathbb{C}^6 and its subspaces

$$\begin{array}{ccccc}
 & & & \langle e_1, e_2, e_3, e_4, e_5 + \lambda e_6 \rangle & \\
 & & & \uparrow & \\
 & & & \langle e_1, e_2, e_3, e_4 \rangle & \\
 & & & \uparrow & \\
 & & & \langle e_2, e_3, e_4 \rangle & \\
 & & & \uparrow & \\
 & & & \langle e_3, e_4 \rangle & \\
 & & & \uparrow & \\
 \langle e_{123}, e_{245}, e_{16} \rangle & \langle e_1, e_2, e_5, e_6 \rangle & & \langle e_5, e_6 \rangle & \langle e_4 \rangle \\
 & \uparrow & & &
 \end{array}$$

- If $\mathcal{S} = (N, 4)$, then $\mathcal{V}_\lambda(\mathcal{S})$ consists of the space \mathbb{C}^5 and its subspaces

$$\begin{array}{ccccc}
 & & & \langle e_1, e_2, e_3, e_4 \rangle & \\
 & & & \uparrow & \\
 & & & \langle e_2, e_3, e_4 \rangle & \\
 & & & \uparrow & \\
 & & & \langle e_3, e_4 \rangle & \\
 & & & \uparrow & \\
 \langle e_{235}, e_{134}, e_5, e_3 + \lambda e_4 \rangle & \langle e_1, e_2, e_5 \rangle & & \langle e_4 \rangle & \\
 \uparrow & \uparrow & \swarrow & & \\
 \langle e_{235}, e_{134} \rangle & \langle e_5 \rangle & & &
 \end{array}$$

Denote by $V_\lambda^\mathcal{S}$ the only subspace from $\mathcal{V}_\lambda(\mathcal{S})$ that depends on the parameter λ and denote by a the element from \mathcal{S} that corresponds to $V_\lambda^\mathcal{S}$. Deleting $V_\lambda^\mathcal{S}$ from $\mathcal{V}_\lambda(\mathcal{S})$, we obtain the subspace representation $\mathcal{V}(\mathcal{S}_a) = (V^\mathcal{S}; V_i^\mathcal{S})_{i \in \mathcal{S}_a}$ of primitive poset $\mathcal{S}_a := \mathcal{S} \setminus \{a\}$.

Proposition 7. $\mathcal{V}_\lambda(\mathcal{S})$ is not equivalent to $\mathcal{V}_\mu(\mathcal{S})$ if $\lambda \neq \mu$ for each critical poset \mathcal{S} . All subspace representation $\mathcal{V}(\mathcal{S}_a)$ are schurian.

Proof. This proposition is proved by straightforward computations. \square

Let \mathcal{S} be a critical poset. The poset \mathcal{S}_a is primitive and does not contain any of the critical posets, its subspace representation $\mathcal{V}(\mathcal{S}_a)$ is schurian. By [3, Proposition 3.1], there exists a weight which we denote by χ^a , such that $\mathcal{V}(\mathcal{S}_a)$ is χ^a -stable. Write

$$R := \min \left\{ \dim M - \sum_{i \in \mathcal{S}_a} \chi_i^a \dim(V_i^{\mathcal{S}} \cap M) \mid M \text{ is a proper subspace of } V^{\mathcal{S}} \right\}.$$

The subspace representation $\mathcal{V}(\mathcal{S}_a)$ is χ^a -stable, hence $R > 0$. Let ε be such that $R > \varepsilon > 0$. Write $T := 1 + (R - \varepsilon) \dim V_{\lambda}^{\mathcal{S}} (\dim V^{\mathcal{S}})^{-1}$ and

$$\chi^{\mathcal{S}} = (\chi_i^{\mathcal{S}})_{i \in \mathcal{S}}, \quad \chi_i^{\mathcal{S}} := \begin{cases} \chi_i^a \cdot T^{-1}, & \text{if } i \in \mathcal{S}_a, \\ (R - \varepsilon) \cdot T^{-1}, & \text{if } i = a. \end{cases}$$

Proposition 8. *The subspace representations $\mathcal{V}_{\lambda}(\mathcal{S})$ are $\chi^{\mathcal{S}}$ -stable for all λ and \mathcal{S} .*

Proof. Note that

$$\begin{aligned} \sum_{i \in \mathcal{S}} \chi_i^{\mathcal{S}} \dim V_i^{\mathcal{S}} &= T^{-1} \sum_{i \in \mathcal{S}_a} \chi_i^a \dim V_i^{\mathcal{S}} + \chi_a^{\mathcal{S}} \dim V_{\lambda}^{\mathcal{S}} \\ &= T^{-1} \dim V^{\mathcal{S}} + (1 - T^{-1}) \dim V^{\mathcal{S}} = \dim V^{\mathcal{S}}. \end{aligned}$$

Let M be any proper subspace of $V^{\mathcal{S}}$. Then

$$\begin{aligned} \sum_{i \in \mathcal{S}} \chi_i^{\mathcal{S}} \dim(V_i^{\mathcal{S}} \cap M) &= T^{-1} \sum_{i \in \mathcal{S}_a} \chi_i^a \dim(V_i^{\mathcal{S}} \cap M) + \chi_a^{\mathcal{S}} \dim(V_{\lambda}^{\mathcal{S}} \cap M) \\ &\leq T^{-1} (\dim M(1 - R) + (R - \varepsilon) \dim(V_{\lambda}^{\mathcal{S}} \cap M)) \\ &< T^{-1} \dim M(1 - \varepsilon) < \dim M. \end{aligned}$$

Hence $\mathcal{V}_{\lambda}(\mathcal{S})$ is $\chi^{\mathcal{S}}$ -stable. \square

Proposition 9. *Critical posets are unitary representation-infinite.*

Proof. By Proposition 7 and Proposition 8, any critical poset \mathcal{S} has an infinite number of nonequivalent $\chi^{\mathcal{S}}$ -stable subspace representations. By Theorem 6, \mathcal{S} has an infinite number of indecomposable unitarily nonequivalent $\chi^{\mathcal{S}}$ -representations. \square

Proposition 10. *If a poset \mathcal{S} contains a critical poset (as a full subposet), then \mathcal{S} is unitary representation-infinite.*

Proof. Suppose that \mathcal{S} contains a critical poset \mathcal{S}_c . By Proposition 9, there exists a weight χ^c such that \mathcal{S}_c has an infinite number of indecomposable unitarily nonequivalent χ^c -representations of dimension d . Define the following subset of \mathcal{S} :

$$\mathcal{S}_{\max} := \{a \in \mathcal{S} \mid b \prec a \text{ for some } b \in \mathcal{S}_c\}.$$

For each χ^c -representation $\mathcal{U} = (U; U_i)_{i \in \mathcal{S}_c}$, define the unitary representation $\mathcal{U}' = (U; U'_i)_{i \in \mathcal{S}}$ of \mathcal{S} as follows:

$$U'_i := \begin{cases} 0, & \text{if } i \notin \mathcal{S}_{\max} \cup \mathcal{S}_c, \\ U_i, & \text{if } i \in \mathcal{S}_c, \\ U, & \text{if } i \in \mathcal{S}_{\max}. \end{cases}$$

It is easy to check that \mathcal{U}' is χ' -representation, in which $\chi' = (\chi'_i)_{i \in \mathcal{S}}$ is defined by

$$\chi'_i := \begin{cases} \chi_i^c \cdot (1 + |\mathcal{S}_{\max}|)^{-1}, & \text{if } i \in \mathcal{S}_c, \\ (1 + |\mathcal{S}_{\max}|)^{-1}, & \text{otherwise.} \end{cases}$$

Hence \mathcal{S} is unitary representation-infinite. \square

The implication (ii) \Rightarrow (iii) follows from Proposition 10. This finishes the proof of Theorem 1.

Remark 11. Define the following weights:

$$\begin{aligned} \chi^{(1,1,1,1)} &:= \frac{1}{2}(1, 1, 1, 1), \\ \chi^{(2,2,2)} &:= \frac{1}{3}(1, 1, 1, 1, 1, 1), \\ \chi^{(1,3,3)} &:= \frac{1}{4}(2, 1, 1, 1, 1, 1, 1), \\ \chi^{(1,2,5)} &:= \frac{1}{6}(3, 2, 2, 1, 1, 1, 1, 1), \\ \chi^{(N,4)} &:= \frac{1}{5}(2, 1, 1, 2, 1, 1, 1, 1). \end{aligned}$$

Each weight $\chi^{\mathcal{S}}$ obtained from the minimal imaginary root of the quadratic form related to a critical poset \mathcal{S} . We checked (describing all possible sub-dimension vectors) that the representations $\mathcal{V}_{\lambda}(\mathcal{S})$ are $\chi^{\mathcal{S}}$ -stable for any $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Hence they give rise to an infinite family of nonequivalent $\chi^{\mathcal{S}}$ -representations. For primitive \mathcal{S} one can obtain the precise description of projections for such representations using the results from [1, 8, 12]. The description in the case $\mathcal{S} = (N, 4)$ is unknown.

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